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Heat transfer analysis for the Falkner–Skan wedge flow by the differential transformation method

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Abstract

This paper presents the differential transformation method to investigate the temperature field associated with the Falkner–Skan boundary-layer problem. A group of transformations are used to reduce the boundary value problem into a pair of initial value problems, which are then solved by means of the differential transformation method. The proposed method yields closed series solutions of a system of the boundary layer equations, which can then be calculated numerically. Numerical results for the dimensionless velocity and temperature profiles of the wedge flow are presented graphically for different values of the wedge angle and Prandtl number. It is seen that the current results are in good agreement with those provided by other numerical methods. Therefore, the method presented in this study provides an effective scheme for determining the solutions of a system of nonlinear boundary-layer problems.

Keywords: Nonlinear; Wedge flow; Thermal boundary-layer problem; Differential transformation method

1. Introduction

A common area of interest in the field of aerodynamics is the analysis of thermal boundary-layer problems for two-dimensional steady and incompressible laminar flow passing a wedge. These types of boundary-layer problems are expressed in the form of nonlinear third-order partial differential equations, which cannot be solved directly in a closed form. Accordingly, it is necessary to develop numerical methods capable of providing accurate solutions for problems of these types. In their pioneering work of 1931, Falkner and Skan [1] considered two-dimensional wedge flows. They developed a similarity transformation method in which the partial differential boundary-layer equation was reduced to a nonlinear third-order ordinary differential equation which could then be solved numerically. In 1979, Na [2] employed a group of transformations to reduce third-order boundary value problem to a pair of initial value problems and then solved these problems by means of a forward integration scheme. In 1983, Rajagopal et al. [3] studied the Falkner–Skan boundary layer flow of a homogeneous incompressible second grade fluid past a wedge placed symmetrically with respect to the flow direction. In 1987, Lin and Lin [4] introduced a similarity solution method for the forced convection heat transfer from isothermal or uniform-flux surfaces to fluids of any Prandtl number. The solutions of the resulting similarity equations are given by the Runge–Kutta scheme. In 1997, Hsu et al. [5] studied the temperature and flow fields of the flow past a wedge by the

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series expansion method, similarity transformation, Runge-Kutta integration and the shooting method. In 1998, Asaithambi [6] presented a finite-difference method for solving the Falkner–Skan equation. Later, Hsu and Hsiao [7] presented a combination of a series expansion, similarity transformation and finite difference method for the heat transfer problem of a second-grade viscoelastic fluid past a plate fin. In our paper [8] employed a combination of the differential transformation method and finite difference approximation to analyze Burgers' equation for various values of Reynolds numbers.

The present study employs the differential transformation method to obtain series solutions of the Falkner–Skan thermal boundary-layer problem. Firstly, a group of transformations are used to reduce the third-order nonlinear boundary value problem to a pair of initial value problems. These problems are then solved by the differential transformation method. The study concludes by comparing the current numerical results with those given by other integral approximation methods in order to verify the accuracy of the proposed method. Although the integral transformation method provides a powerful technique to solve linear differential equations, it is not so easily applied to the solution of nonlinear differential equations. The differential transformation method is better suited to solving this type of equation, and is, therefore, the method that is adopted within this present study. The differential transformation method consists of three basic steps: (1) the differential equations are transformed into algebraic equations, (2) these algebraic equations are solved, and (3) a process of inverse transformation is applied to determine the solution of the given problem. The differential transformation method yields a power series, close-form solution, and has the advantage that nonlinear differential equations may be solved directly, i.e. without the need for iterative calculations.

2. Mathematical analysis

Consider the flow of an incompressible viscous fluid over a wedge, as shown in Fig. 1. The temperature of the wall, T_w, is uniform and constant and is greater than the free stream temperature, T_∞. It is assumed that the free stream velocity, U_∞, is also uniform and constant. Further, assuming that the flow in the laminar boundary layer is two-dimensional, and that the temperature changes resulting from viscous dissipation are small, the continuity equation and the boundary-layer equations may be expressed as

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U\frac{\mathrm{d}U}{\mathrm{d}x}+v\frac{\partial^2 u}{\partial y^2}, \tag{2}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}, \tag{3}$$

where u and v are the respective velocity components in the x- and y-direction of the fluid flow, v is the viscosity of the fluid, and U is the reference velocity at the edge of the boundary layer and is a function of x. α is the thermal diffusivity of the fluid, T is the temperature in the vicinity of the wedge, and the boundary conditions are given by

$$\text{at } y=0: u=v=0, \quad \text{and} \quad T=T_w, \tag{4}$$

$$\text{as } y\to\infty: u\to U(x)=U_\infty(x/L)^m, \quad \text{and} \quad T=T_\infty, \tag{5}$$

$$\text{at } x=0: u=U_\infty \quad \text{and} \quad T=T_\infty, \tag{6}$$

where U_∞ is the meanstream velocity, L is the length of the wedge, m is the Falkner–Skan power-law parameter, and x is measured from the tip of the wedge. A stream function, $\Psi(x,y)$, is introduced such that

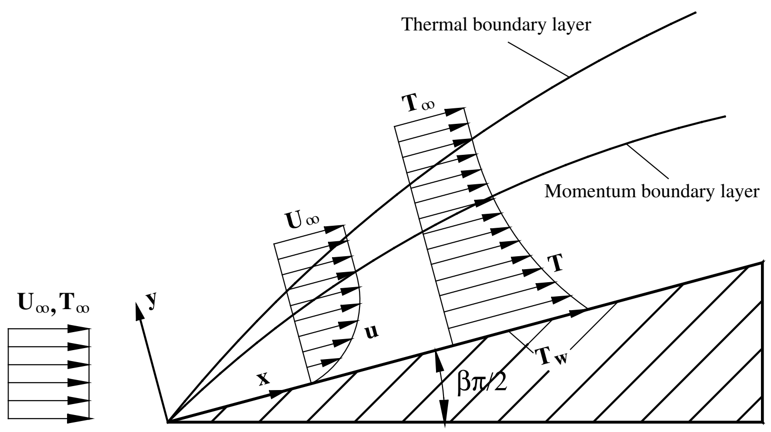

Fig. 1. Velocity and thermal boundary layers for the Falkner–Skan wedge flow.

$$u = \frac{\partial \Psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \Psi}{\partial x}. \tag{7}$$

In addition to the physical considerations which require the introduction of this function, the mathematical significance of its use is that the equation of continuity, i.e. Eq. (1), is satisfied identically. The momentum equation becomes:

$$\frac{\partial \Psi}{\partial y}\frac{\partial^2 \Psi}{\partial x \partial y} - \frac{\partial \Psi}{\partial x}\frac{\partial^2 \Psi}{\partial y^2} = U\frac{\mathrm{d}U}{\mathrm{d}x} + v\frac{\partial^3 \Psi}{\partial y^3}. \tag{8}$$

Integrating Eq. (7) and introducing a similarity variable yields:

$$f(\eta) = \sqrt{\frac{1+m}{2}\frac{L^m}{vU_\infty}} \cdot (\Psi/x^{(1+m)/2}), \tag{9}$$

$$\eta = \sqrt{\frac{1+m}{2}\frac{U_\infty}{vL^m}} \cdot (y/x^{(1-m)/2}). \tag{10}$$

Substituting Eqs. (9) and (10) into Eq. (8) gives

$$\frac{\mathrm{d}^3 f(\eta)}{\mathrm{d}\eta^3} + f(\eta)\frac{\mathrm{d}^2 f(\eta)}{\mathrm{d}\eta^2} + \beta\left[1 - \left(\frac{\mathrm{d}f(\eta)}{\mathrm{d}\eta}\right)^2\right] = 0, \tag{11}$$

which is known as the Falkner–Skan boundary-layer equation [1]. The boundary conditions of $f(\eta)$ are given by

$$\text{at } \eta = 0: \quad f(0) = \frac{\mathrm{d}f(0)}{\mathrm{d}\eta} = 0, \tag{12}$$

$$\text{as } \eta \to \infty: \quad \frac{\mathrm{d}f(\infty)}{\mathrm{d}\eta} = 1. \tag{13}$$

Note that in the equations above, parameters β and m are related through the expression $\beta = 2m/(1+m)$. A dimensionless temperature is defined as follows:

$$\theta = \frac{T - T_w}{T_\infty - T_w}. \tag{14}$$

If Eq. (14) is substituted into Eq. (3), the boundary-layer energy equation then becomes:

$$\frac{\mathrm{d}^2\theta(\eta)}{\mathrm{d}\eta^2} + Pr \cdot f(\eta,\beta) \cdot \frac{\mathrm{d}\theta(\eta)}{\mathrm{d}\eta} = 0, \tag{15}$$

with the following boundary conditions:

$$\text{at } \eta = 0: \quad \theta = 0, \tag{16}$$

$$\text{at } \eta \to \infty: \quad \theta = 1, \tag{17}$$

where Pr is the Prandtl number, which is equal to the ratio of the momentum diffusivity of the fluid to its thermal diffusivity (i.e. $Pr = v/\alpha$). Eqs. (11) and (15) present a system of ordinary differential equations for the Falkner–Skan boundary-layer problem. Simultaneous solution of these two equations yields the velocity and temperature profiles for the flow of a viscous fluid passing a wedge. In order to solve the Falkner–Skan boundary-layer equation for a family of values of β, it is first necessary to define a dependent variable, $g(\eta)$, i.e.

$$g(\eta) = \frac{\partial f(\eta)}{\partial \beta}. \tag{18}$$

Differentiating Eqs. (11)–(13) with respect to β gives

$$\frac{\mathrm{d}^3 g(\eta)}{\mathrm{d}\eta^3} + f(\eta)\frac{\mathrm{d}^2 g(\eta)}{\mathrm{d}\eta^2} + g(\eta)\frac{\mathrm{d}^2 f(\eta)}{\mathrm{d}\eta^2} + \left[1 - \left(\frac{\mathrm{d}f(\eta)}{\mathrm{d}\eta}\right)^2\right] - 2\beta\frac{\mathrm{d}f(\eta)}{\mathrm{d}\eta}\frac{\mathrm{d}g(\eta)}{\mathrm{d}\eta} = 0. \tag{19}$$

The boundary conditions are given by

$$g(0) = \frac{\mathrm{d}g(0)}{\mathrm{d}\eta} = 0, \quad \frac{\mathrm{d}g(\infty)}{\mathrm{d}\eta} = 0. \tag{20}$$

The method of superposition is used together with a group of transformations to solve the boundary-layer equation given in Eq. (19). Initially, the following expression is defined:

$$g(\eta) = P(\eta) + C_1 \cdot Q(\eta), \tag{21}$$

where C_1 is a constant to be determined.

Substituting Eq. (21) into Eq. (19) gives the following pair of initial value problems:

$$\frac{\mathrm{d}^3 P(\eta)}{\mathrm{d}\eta^3} + f(\eta)\frac{\mathrm{d}^2 P(\eta)}{\mathrm{d}\eta^2} + P(\eta)\frac{\mathrm{d}^2 f(\eta)}{\mathrm{d}\eta^2} - 2\beta\frac{\mathrm{d}f(\eta)}{\mathrm{d}\eta}\frac{\mathrm{d}P(\eta)}{\mathrm{d}\eta} = \left(\frac{\mathrm{d}f(\eta)}{\mathrm{d}\eta}\right)^2 - 1, \tag{22}$$

with initial conditions of

$$P(0) = \frac{\mathrm{d}P(0)}{\mathrm{d}\eta} = \frac{\mathrm{d}^2 P(0)}{\mathrm{d}\eta^2} = 0, \tag{23}$$

and

$$\frac{\mathrm{d}^3 Q(\eta)}{\mathrm{d}\eta^3} + f(\eta)\frac{\mathrm{d}^2 Q(\eta)}{\mathrm{d}\eta^2} + Q(\eta)\frac{\mathrm{d}^2 f(\eta)}{\mathrm{d}\eta^2} - 2\beta\frac{\mathrm{d}f(\eta)}{\mathrm{d}\eta}\frac{\mathrm{d}Q(\eta)}{\mathrm{d}\eta} = 0, \tag{24}$$

with initial conditions of

$$Q(0) = \frac{\mathrm{d}Q(0)}{\mathrm{d}\eta} = 0, \quad \frac{\mathrm{d}^2 Q(0)}{\mathrm{d}\eta^2} = 1. \tag{25}$$

Substituting the boundary condition at infinity from Eq. (20) into Eq. (21) gives the value of the parameter C_1 as

$$C_1 = -\frac{\mathrm{d}P(\infty)/\mathrm{d}\eta}{\mathrm{d}Q(\infty)/\mathrm{d}\eta}. \tag{26}$$

To solve Eq. (11) at $\beta = \Delta\beta$, Eq. (11) is first solved for the case of $\beta = 0$ in order to establish the function $f(\eta)$ and its derivatives which appear in Eqs. (22) and (24). Solving Eqs. (22)–(25) then gives $P(\eta)$, $Q(\eta)$, and their derivatives. The value of C_1 is obtained by substituting $\mathrm{d}P(\infty)/\mathrm{d}\eta$ and $\mathrm{d}Q(\infty)/\mathrm{d}\eta$ into Eq. (26). Given C_1, the values of $g(\eta)$ are derived from Eq. (21) and are then substituted into the rearranged form of Eq. (18) given below to give the solutions of $f(\eta)$ at $\beta = \Delta\beta$, i.e.

$$f(\eta)|_{\beta=\Delta\beta} = f(\eta)|_{\beta=0} + g(\eta)\cdot\Delta\beta. \tag{27}$$

This process is then repeated to calculate the solutions of Eq. (11) for $\beta = 2\Delta\beta,\ \beta = 3\Delta\beta,\ldots$, etc.

Since the solutions of Eq. (11) for the various values of β can be established from the previous calculations, $f(\eta)$ is also known and can be substituted into Eq. (15) to solve the boundary-layer energy equation. Eq. (15) is a linear second-order ordinary differential equation with variable coefficients. The solution of this energy equation can be obtained by using the method of superposition. The following relationship is established:

$$\theta(\eta) = C(\eta) + C_2\cdot D(\eta). \tag{28}$$

Substituting Eq. (28) into Eq. (15) and separating the resulting equations into a group of terms, gives two initial value problems, i.e.

$$\frac{d^2C(\eta)}{d\eta^2} + Pr\cdot f(\eta,\beta)\cdot\frac{dC(\eta)}{d\eta} = 0, \tag{29}$$

with initial conditions of

$$\eta = 0:\quad C(0) = 0,\quad \frac{dC(0)}{d\eta} = 1, \tag{30}$$

and

$$\frac{d^2D(\eta)}{d\eta^2} + Pr\cdot f(\eta,\beta)\cdot\frac{dD(\eta)}{d\eta} = 0, \tag{31}$$

with initial conditions of

$$\eta = 0:\quad D(0) = 0,\quad \frac{dD(0)}{d\eta} = -1. \tag{32}$$

Substituting Eqs. (30) and (32) into Eq. (28) gives

$$\frac{d\theta(0)}{d\eta} = 1 - C_2. \tag{33}$$

The parameter "C_2" in Eq. (28) can be calculated by using the boundary condition given in Eq. (17). This yields

$$C_2 = \frac{1 - C(\infty)}{D(\infty)}. \tag{34}$$

By solving Eqs. (29)–(32) then gives $C(\eta)$, $D(\eta)$, and their derivatives. The value of C_2, the values of $\theta(\eta)$ are derived from Eq. (28). Hence, we have been determined the solutions of the Falkner–Skan wedge flow.

3. Numerical formulation—differential transformation method

To solve Eq. (11) using the differential transformation method, it is first necessary to solve the Blasius equation ($\beta = 0$), i.e.

$$\frac{d^3f(\eta)}{d\eta^3} + f(\eta)\frac{d^2f(\eta)}{d\eta^2} = 0. \tag{35}$$

The boundary conditions are given by

$$\text{at } \eta = 0:\quad f(0) = \frac{df(0)}{d\eta} = 0, \tag{36}$$

$$\text{as } \eta\to\infty:\quad \frac{df(\infty)}{d\eta} = 1. \tag{37}$$

The boundary value problems (Eqs. (35)–(37)) can then be reduced to a pair of initial value problems, which are given by

$$\frac{d^3F(\xi)}{d\xi^3} + F(\xi)\frac{d^2F(\xi)}{d\xi^2} = 0, \tag{38}$$

with initial conditions of

$$\xi = 0:\quad F(0) = \frac{dF(0)}{d\xi} = 0,\quad \frac{d^2F(0)}{d\xi^2} = 1, \tag{39}$$

and by

$$\frac{d^3f(\eta)}{d\eta^3} + f(\eta)\frac{d^2f(\eta)}{d\eta^2} = 0, \tag{40}$$

with initial conditions of

$$\eta = 0:\quad f(0) = \frac{df(0)}{d\eta} = 0,\quad \frac{d^2f(0)}{d\eta^2} = \left[\frac{1}{dF(\infty)/d\xi}\right]^{3/2}. \tag{41}$$

These equations suggest a transformation of the form:

$$F(\xi) = \lambda^{-1/3}f(\eta),\quad \xi = \lambda^{1/3}\eta,\quad \lambda = \left[\frac{1}{dF(\infty)/d\xi}\right]^{3/2}. \tag{42}$$

The differential transformation method is then used to solve the pair of initial value problems (Eqs. (38)–(41)). Initially, the following expressions are defined:

$$y(\xi) = \frac{dF(\xi)}{d\xi}, \tag{43}$$

and

$$z(\xi) = \frac{dy(\xi)}{d\xi} = \frac{d^2F(\xi)}{d\xi^2}. \tag{44}$$

Thereafter, the third-order ordinary differential equation (Eq. (38)) is reduced to a first-order ordinary differential equation with the following form:

$$\frac{dz(\xi)}{d\xi} + F(\xi)\cdot z(\xi) = 0. \tag{45}$$

The initial conditions become

$$\xi = 0:\quad F(0) = y(0) = 0,\quad z(0) = 1. \tag{46}$$

By a process of inverse differential transformation, the solutions of each sub-domain take $m+1$ terms for the power series, i.e.

$$F_i(\xi) = \sum_{k=0}^{m} \left(\frac{\xi}{H_i}\right)^k \overline{F}_i(k), \quad 0 \leqslant \xi \leqslant H_i, \tag{47}$$

$$y_i(\xi) = \sum_{k=0}^{m} \left(\frac{\xi}{H_i}\right)^k Y_i(k), \quad 0 \leqslant \xi \leqslant H_i, \tag{48}$$

$$z_i(\xi) = \sum_{k=0}^{m} \left(\frac{\xi}{H_i}\right)^k Z_i(k), \quad 0 \leqslant \xi \leqslant H_i, \tag{49}$$

where $i = 0, 1, 2, \ldots, n$ indicates the ith sub-domain, $k = 0, 1, 2, \ldots, m$ represents the number of terms of the power series, H_i represents the sub-domain interval, and $\overline{F}_i(k)$, $Y_i(k)$ and $Z_i(k)$ are the transformed functions of $F_i(\xi)$, $y_i(\xi)$ and $z_i(\xi)$, respectively. From the initial conditions (Eq. (46)) and the solution equations (Eqs. (47)–(49)), it can be seen that

$$\overline{F}_0(0) = 0, \tag{50}$$

$$Y_0(0) = 0, \tag{51}$$

$$Z_0(0) = \delta(k), \quad \text{where } \delta(k) = \begin{cases} 1 & k = 0, \\ 0 & k \neq 0. \end{cases} \tag{52}$$

Performing differential transformation of Eqs. (43)–(45) gives the following:

$$\frac{k+1}{H_i}\overline{F}_i(k+1) = Y_i(k), \tag{53}$$

$$\frac{k+1}{H_i}Y_i(k+1) = Z_i(k), \tag{54}$$

$$\frac{k+1}{H_i}Z_i(k+1) + \overline{F}_i(k) * Z_i(k) = \frac{k+1}{H_i}Z_i(k+1) + \sum_{l=0}^{k}\overline{F}_i(k-l)Z_i(l) = 0. \tag{55}$$

The various values of $\overline{F}_i(k)$, $Y_i(k)$ and $Z_i(k)$ are obtained by using Eqs. (53)–(55), together with the transformed initial conditions, i.e. Eqs. (50)–(52). The solution of Eq. (38) is then determined by means of the inverse transformed equations, i.e. Eqs. (47)–(49).

From Eq. (41), it can be shown that the value of $\mathrm{d}F(\infty)/\mathrm{d}\xi$ approaches a limiting value in the final sub-domain. In the expressions which follow, this limiting value is represented by the parameter "λ". The following expressions are also defined:

$$u(\eta) = \frac{\mathrm{d}f(\eta)}{\mathrm{d}\eta}, \tag{56}$$

and

$$v(\eta) = \frac{\mathrm{d}u(\eta)}{\mathrm{d}\eta} = \frac{\mathrm{d}^2 f(\eta)}{\mathrm{d}\eta^2}. \tag{57}$$

Hence, the third-order ordinary differential equation (Eq. (40)) becomes a first-order ordinary differential equation with the following form:

$$\frac{\mathrm{d}v(\eta)}{\mathrm{d}\eta} + f(\eta) \cdot v(\eta) = 0. \tag{58}$$

The initial conditions become

$$\eta = 0: \quad f(0) = u(0) = 0, \quad v(0) = \lambda^{-3/2}. \tag{59}$$

As in the previous procedure, inverse differential transformation is used to yield the following solutions:

$$f_i(\eta) = \sum_{k=0}^{m} \left(\frac{\eta}{H_i}\right)^k \bar{f}_i(k), \quad 0 \leqslant \eta \leqslant H_i, \tag{60}$$

$$u_i(\eta) = \sum_{k=0}^{m} \left(\frac{\eta}{H_i}\right)^k U_i(k), \quad 0 \leqslant \eta \leqslant H_i, \tag{61}$$

$$v_i(\eta) = \sum_{k=0}^{m} \left(\frac{\eta}{H_i}\right)^k V_i(k), \quad 0 \leqslant \eta \leqslant H_i, \tag{62}$$

where, as before, $i = 0, 1, 2, \ldots, n$ indicates the ith sub-domain, $k = 0, 1, 2, \ldots, m$ represents the number of terms of the power series, H_i represents the sub-domain interval, and $\bar{f}_i(k)$, $U_i(k)$ and $V_i(k)$ are the transformed functions of $f_i(\eta)$, $u_i(\eta)$ and $v_i(\eta)$, respectively. From the initial conditions (Eq. (59)) and the solution equations (Eqs. (60)–(62)), it can be shown that

$$\bar{f}_0(0) = 0, \tag{63}$$

$$U_0(0) = 0, \tag{64}$$

$$V_0(0) = \lambda^{-3/2} \cdot \delta(k), \quad \text{where } \delta(k) = \begin{cases} 1 & k = 0, \\ 0 & k \neq 0. \end{cases} \tag{65}$$

Eqs. (56)–(58) undergo a process of differential transformation to give the following:

$$\frac{k+1}{H_i}\bar{f}_i(k+1) = U_i(k), \tag{66}$$

$$\frac{k+1}{H_i}U_i(k+1) = V_i(k), \tag{67}$$

$$\frac{k+1}{H_i}V_i(k+1) + \bar{f}_i(k) * V_i(k) = \frac{k+1}{H_i}V_i(k+1) + \sum_{l=0}^{k}\bar{f}_i(k-l)V_i(l) = 0. \tag{68}$$

As in the solution of the previous initial value problem, when the various values of $\bar{f}_i(k)$, $U_i(k)$ and $V_i(k)$ have been determined by using Eqs. (66)–(68), together with the transformed initial conditions (Eqs. (63)–(65)), the solution of Eq. (40) can be obtained by means of the inverse transformed equations, i.e. Eqs. (60)–(62).

Since the solutions of the boundary value problems (Eqs. (35)–(37)) can be established from the previous calculations, $f(\eta)$ is also known and can be substituted into Eq. (11) to solve the Falkner–Skan equation.

The differential transformation method is then used to solve the pair of initial value problems given by Eqs. (22)–(25). Initially, the following expressions are defined:

$$r(\eta) = \frac{\mathrm{d}P(\eta)}{\mathrm{d}\eta}, \tag{69}$$

and

$$s(\eta) = \frac{\mathrm{d}r(\eta)}{\mathrm{d}\eta} = \frac{\mathrm{d}^2 P(\eta)}{\mathrm{d}\eta^2}. \tag{70}$$

Thereafter, the third-order ordinary differential equation (Eq. (22)) is reduced to a first-order ordinary differential equation with the following form:

$$\frac{\mathrm{d}s(\eta)}{\mathrm{d}\eta} + f(\eta)\cdot s(\eta) + \frac{\mathrm{d}^2 f(\eta)}{\mathrm{d}\eta^2}\cdot P(\eta) - 2\beta\frac{\mathrm{d}f(\eta)}{\mathrm{d}\eta}\cdot r(\eta) = \left(\frac{\mathrm{d}f(\eta)}{\mathrm{d}\eta}\right)^2 - 1. \tag{71}$$

The initial conditions become

$$\eta = 0: \quad P(0) = r(0) = s(0) = 0. \tag{72}$$

As before, inverse differential transformation is used to yield the following solutions:

$$P_i(\eta) = \sum_{k=0}^{m}\left(\frac{\eta}{H_i}\right)^k \overline{P}_i(k), \quad 0 \leqslant \eta \leqslant H_i, \tag{73}$$

$$r_i(\eta) = \sum_{k=0}^{m}\left(\frac{\eta}{H_i}\right)^k R_i(k), \quad 0 \leqslant \eta \leqslant H_i, \tag{74}$$

$$s_i(\eta) = \sum_{k=0}^{m}\left(\frac{\eta}{H_i}\right)^k S_i(k), \quad 0 \leqslant \eta \leqslant H_i, \tag{75}$$

where $i = 0, 1, 2, \ldots, n$ indicates the ith sub-domain, $k = 0, 1, 2, \ldots, m$ represents the number of terms of the power series, H_i represents the sub-domain interval, and $\overline{P}_i(k)$, $R_i(k)$ and $S_i(k)$ are the transformed functions of $P_i(\eta)$, $r_i(\eta)$ and $s_i(\eta)$, respectively. From the initial conditions (Eq. (72)) and the solution equations (Eqs. (73)–(75)), it can be shown that

$$\overline{P}_0(0) = 0, \tag{76}$$

$$R_0(0) = 0, \tag{77}$$

$$S_0(0) = 0. \tag{78}$$

Eqs. (69)–(71) undergo a process of differential transformation to give the following:

$$\frac{k+1}{H_i}\overline{P}_i(k+1) = R_i(k), \tag{79}$$

$$\frac{k+1}{H_i}R_i(k+1) = S_i(k), \tag{80}$$

$$\frac{k+1}{H_i}S_i(k+1) + \sum_{l=0}^{k}\bar{f}_i(k-l)\cdot S_i(l) + \sum_{l=0}^{k}Z_i(k-l)\cdot\overline{P}_i(l) - 2\beta\sum_{l=0}^{k}Y(k-l)\cdot R(l) = \sum_{l=0}^{k}Y(k-l)\cdot Y(l) - \delta_i(k). \tag{81}$$

As in the solution of the previous initial value problem, when the various values of $\overline{P}_i(k)$, $R_i(k)$ and $S_i(k)$ have been determined by using Eqs. (79)–(81), together with the transformed initial conditions (Eqs. (76)–(78)), the solution of Eq. (71) can be obtained by means of the inverse transformed equations, i.e. Eqs. (73)–(75). From Eq. (74), it is noted that the value of $\mathrm{d}P(\infty)/\mathrm{d}\eta$ approaches a limiting value in the final sub-domain. The following expression is defined:

$$A(\eta) = \frac{\mathrm{d}Q(\eta)}{\mathrm{d}\eta}, \tag{82}$$

and

$$B(\eta) = \frac{\mathrm{d}A(\eta)}{\mathrm{d}\eta} = \frac{\mathrm{d}^2 Q(\eta)}{\mathrm{d}\eta^2}. \tag{83}$$

Thereafter, the third-order ordinary differential equation (Eq. (24)) is reduced to a first-order ordinary differential equation with the following form:

$$\frac{\mathrm{d}B(\eta)}{\mathrm{d}\eta} + f(\eta)\cdot B(\eta) + \frac{\mathrm{d}^2 f(\eta)}{\mathrm{d}\eta^2}\cdot Q(\eta) - 2\beta\frac{\mathrm{d}f(\eta)}{\mathrm{d}\eta}\cdot A(\eta) = 0. \tag{84}$$

The initial conditions become

$$\eta = 0: \quad Q(0) = A(0) = B(0) = 0. \tag{85}$$

Inverse differential transformation is again used to yield the following solutions:

$$Q_i(\eta) = \sum_{k=0}^{m}\left(\frac{\eta}{H_i}\right)^k \overline{Q}_i(k), \quad 0 \leqslant \eta \leqslant H_i, \tag{86}$$

$$A_i(\eta) = \sum_{k=0}^{m}\left(\frac{\eta}{H_i}\right)^k \overline{A}_i(k), \quad 0 \leqslant \eta \leqslant H_i, \tag{87}$$

$$B_i(\eta) = \sum_{k=0}^{m}\left(\frac{\eta}{H_i}\right)^k \overline{B}_i(k), \quad 0 \leqslant \eta \leqslant H_i, \tag{88}$$

where $i = 0, 1, 2, \ldots, n$ indicates the ith sub-domain, $k = 0, 1, 2, \ldots, m$ represents the number of terms of the power series, H_i represents the sub-domain interval, and $\overline{Q}_i(k)$, $\overline{A}_i(k)$ and $\overline{B}_i(k)$ are the transformed functions of $Q_i(\eta)$, $A_i(\eta)$ and $B_i(\eta)$, respectively. From the initial conditions (Eq. (85)) and the solution equations (Eqs. (86)–(88)), it can be shown that

$$\overline{Q}_0(0) = 0, \tag{89}$$

$$\overline{A}_0(0) = 0, \tag{90}$$

$$\overline{B}_0(0) = 0. \tag{91}$$

Eqs. (82)–(84) undergo a process of differential transformation to give the following:

$$\frac{k+1}{H_i}\overline{Q}_i(k+1) = \overline{A}_i(k), \tag{92}$$

$$\frac{k+1}{H_i}\overline{A}_i(k+1) = \overline{B}_i(k), \tag{93}$$

$$\frac{k+1}{H_i}\overline{B}_i(k+1) + \sum_{l=0}^{k}\bar{f}_i(k-l)\cdot\overline{B}_i(l) + \sum_{l=0}^{k}Z_i(k-l)\cdot\overline{Q}_i(l) - 2\beta\sum_{l=0}^{k}Y(k-l)\cdot\overline{A}(l) = \sum_{l=0}^{k}Y(k-l)\cdot Y(l) - \delta_i(k). \tag{94}$$

As in the solution of the previous initial value problem, when the various values of $\overline{Q}_i(k)$, $\overline{A}_i(k)$ and $\overline{B}_i(k)$ have been determined by using Eqs. (92)–(94), together with the transformed initial conditions (Eqs. (89)–(91)), the solution of Eq. (84) can be obtained by means of the inverse transformed equations, i.e. Eqs. (86)–(88). From Eq. (87), it can be seen that the value of $\mathrm{d}Q(\infty)/\mathrm{d}\eta$ approaches a limiting value in the final sub-domain.

The value of C_1 is determined by substituting the values of $\mathrm{d}P(\infty)/\mathrm{d}\eta$ and $\mathrm{d}Q(\infty)/\mathrm{d}\eta$ into Eq. (26). C_1, $P(\eta)$, $Q(\eta)$ and their derivatives are then substituted into Eq. (21) to determine the value of $g(\eta)$. Finally, $g(\eta)$ is substituted into Eq. (27) to generate the solutions of the Falkner–Skan boundary layer equation for various values of β.

The solutions of the pair of linear second-order ordinary differential equations (Eqs. (29)–(32)) can be obtained from the differential transformation method. Initially, the following relationship is defined:

$$w(\eta) = \frac{\mathrm{d}C(\eta)}{\mathrm{d}\eta}. \tag{95}$$

Substituting Eq. (95) into Eq. (29) gives

$$\frac{\mathrm{d}w(\eta)}{\mathrm{d}\eta} + Pr\cdot f(\eta,\beta)\cdot w(\eta) = 0. \tag{96}$$

The initial conditions become

$$\eta = 0: \quad C(0) = 0, \quad w(0) = 1. \tag{97}$$

By a process of inverse differential transformation, the solutions of each sub-domain take $m+1$ terms for the power series, i.e.

$$C_i(\eta) = \sum_{k=0}^{m}\left(\frac{\eta}{H_i}\right)^k\overline{C}_i(k), \quad 0 \leqslant \eta \leqslant H_i, \tag{98}$$

$$w_i(\eta) = \sum_{k=0}^{m}\left(\frac{\eta}{H_i}\right)^k W_i(k), \quad 0 \leqslant \eta \leqslant H_i, \tag{99}$$

where $\overline{C}_i(k)$ and $W_i(k)$ are the transformed functions of $C_i(\eta)$ and $w_i(\eta)$, respectively. From the initial conditions (Eq. (97)) and the solution equations (Eqs. (98) and (99)), it can be shown that

$$\overline{C}_0(0) = 0, \tag{100}$$

$$W_0(0) = \delta(k), \quad \text{where } \delta(k) = \begin{cases} 1 & k = 0, \\ 0 & k \neq 0. \end{cases} \tag{101}$$

Eqs. (95) and (96) undergo the differential transformation to give the following:

$$\frac{k+1}{H_i}\overline{C}_i(k+1) = W_i(k), \tag{102}$$

$$\frac{k+1}{H_i}W_i(k+1) + \bar{f}_i(k) * W_i(k) = \frac{k+1}{H_i}W_i(k+1) + \sum_{l=0}^{k}\bar{f}_i(k-l)W_i(l) = 0. \tag{103}$$

When the various values of $\overline{C}_i(k)$ and $W_i(k)$ have been obtained by using Eqs. (102) and (103), together with the transformed initial conditions (Eqs. (100) and (101)), the solution of Eq. (29) can be determined by using the inverse transformed equations, i.e. Eqs. (98) and (99). From Eq. (98), it is noted that the value of $C(\infty)$ approaches a limiting value in the final sub-domain. The following expression is established:

$$x(\eta) = \frac{\mathrm{d}D(\eta)}{\mathrm{d}\eta}. \tag{104}$$

Substituting Eq. (104) into Eq. (31) yields

$$\frac{\mathrm{d}^2x(\eta)}{\mathrm{d}\eta^2} + Pr\cdot f(\eta,\beta)\cdot\frac{\mathrm{d}x(\eta)}{\mathrm{d}\eta} = 0. \tag{105}$$

The initial conditions become

$$\eta = 0: \quad D(0) = 0, \quad x(0) = 1. \tag{106}$$

By a process of inverse differential transformation, the solutions of each sub-domain take $m+1$ terms for the power series, i.e.

$$D_i(\eta) = \sum_{k=0}^{m}\left(\frac{\eta}{H_i}\right)^k\overline{D}_i(k), \quad 0 \leqslant \eta \leqslant H_i, \tag{107}$$

$$x_i(\eta) = \sum_{k=0}^{m}\left(\frac{\eta}{H_i}\right)^k X_i(k), \quad 0 \leqslant \eta \leqslant H_i, \tag{108}$$

where $\overline{D}_i(k)$ and $X_i(k)$ are the transformed functions of $D_i(\eta)$ and $x_i(\eta)$, respectively. From the initial conditions (Eq. (106)) and the solution equations (Eqs. (107) and (108)), it is shown that

$$\overline{D}_0(0) = 0, \tag{109}$$

$$X_0(0) = \delta(k), \quad \text{where } \delta(k) = \begin{cases} 1 & k = 0, \\ 0 & k \neq 0. \end{cases} \tag{110}$$

Eqs. (104) and (105) undergo the differential transformation to give the following:

$$\frac{k+1}{H_i}\overline{D}_i(k+1) = X_i(k), \tag{111}$$

$$\frac{k+1}{H_i}X_i(k+1) + \bar{f}_i(k) * X_i(k) = \frac{k+1}{H_i}X_i(k+1) + \sum_{l=0}^{k}\bar{f}_i(k-l)X_i(l) = 0. \tag{112}$$

When the various values of $\overline{D}_i(k)$ and $X_i(k)$ have been obtained by using Eqs. (111) and (112), together with the transformed initial conditions (Eqs. (109) and (110)), the solution of Eq. (31) can be obtained from the inverse transformed equations, i.e. Eqs. (107) and (108). From Eq. (107), it is established that the value of $D(\infty)$ approaches a limiting value in the final subdomain.

4. Numerical results and discussion

The present study uses the differential transformation method described above to generate a series of numerical results for the thermal boundary-layer problem in the case of a two-dimensional incompressible flow passing over a wedge. By solving the initial value problem (Eqs. (38) and (39)), it can be seen that $\mathrm{d}F(\infty)/\mathrm{d}\xi$ approaches a limiting value of 1.655190. Substituting the value of $\mathrm{d}F(\infty)/\mathrm{d}\xi$ into Eq. (42) gives a calculated value of λ equal to 0.469600. Using this value of λ, Table 1 presents the current numerical results for the Falkner–Skan boundary-layer problem for the case of $\beta = 0$ in terms of $f(\eta)$ and its derivatives. From Table 1, the results obtained by the present method are in good agreement with those provided by White [9] to about 6 decimal places. Fig. 2 plots the variation in the values of $f(\eta)$ and its derivatives for various values of β. The

Table 1
Results of the Falkner–Skan boundary-layer equation for the case of $\beta = 0$

η	$f(\eta)$		$f'(\eta)$		$f''(\eta)$	
	Present	White [9]	Present	White [9]	Present	White [9]
.00	.000000	.00000	.000000	.00000	.469600	.46960
.10	.002348	.00235	.046959	.04696	.469563	.46956
.20	.009391	.00939	.093905	.09391	.469306	.46931
.30	.021128	.02113	.140806	.14081	.468609	.46861
.40	.037549	.03755	.187605	.18761	.467254	.46725
.50	.058643	.05864	.234228	.23423	.465030	.46503
.60	.084386	.08439	.280575	.28058	.461734	.46173
.70	.114745	.11474	.326532	.32653	.457178	.45718
.80	.149674	.14967	.371963	.37196	.451190	.45119
.90	.189115	.18911	.416718	.41672	.443628	.44363
1.00	.232990	.23299	.460633	.46063	.434379	.43438
1.10	.281208	.28121	.503535	.50354	.423369	.42337
1.20	.333657	.33366	.545246	.54525	.410565	.41057
1.30	.390211	.39021	.585589	.58559	.395985	.39598
1.40	.450724	.45072	.624386	.62439	.379692	.37969
1.50	.515031	.51503	.661474	.66147	.361804	.36180
1.60	.582956	.58296	.696700	.69670	.342487	.34249
1.70	.654305	.65430	.729931	.72993	.321951	.32195
1.80	.728872	.72887	.761057	.76106	.300445	.30045
1.90	.806443	.80644	.789997	.79000	.278251	.27825
2.00	.886797	.88680	.816695	.81669	.255669	.25567
2.20	1.054947	1.05495	.863304	.86330	.210580	.21058
2.40	1.231528	1.23153	.901065	.90107	.167560	.16756
2.60	1.414824	1.41482	.930601	.93060	.128613	.12861
2.80	1.603284	1.60328	.952875	.95288	.095113	.09511
3.00	1.795568	1.79557	.969055	.96905	.067710	.06771
3.20	1.990581	1.99058	.980365	.98037	.046370	.04637
3.40	2.187467	2.18747	.987970	.98797	.030535	.03054
3.60	2.385590	2.38559	.992888	.99289	.019329	.01933
3.80	2.584499	2.58450	.995944	.99594	.011759	.01176
4.00	2.783886	2.78388	.997770	.99777	.006874	.00687
4.20	2.983555	2.98355	.998818	.99882	.003861	.00386
4.40	3.183383	3.18338	.999397	.99940	.002084	.00208
4.60	3.383296	3.38329	.999703	.99970	.001081	.00108
4.80	3.583254	3.58325	.999859	.99986	.000538	.00054
5.00	3.783235	3.78323	.999936	.99994	.000258	.00026

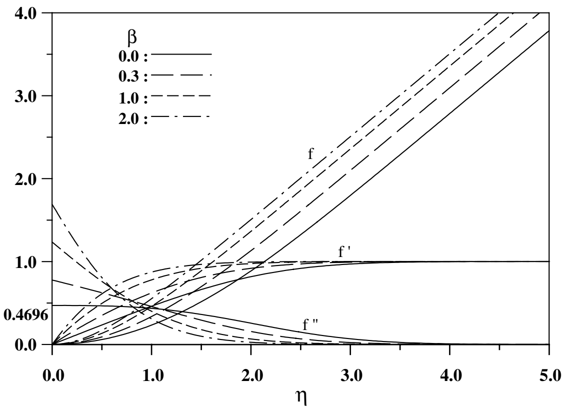

Fig. 2. Numerical results of $f(\eta)$ and its derivatives for various values of β.

Table 2
Comparison of the results for the laminar boundary layer over a wedge

β	$f''(0)$	
	Present method	Rajagopal et al. [3]
.00	0.469600	0.469600
.05	0.531725	0.531130
.10	0.587889	0.587035
.20	0.687641	0.686708
.30	0.775524	0.774755
.40	0.854937	0.854421
.50	0.927906	0.927680
.60	0.995758	0.995836
.70	1.059421	–
.80	1.119574	1.120268
.90	1.176730	–
1.00	1.231289	1.232585
1.20	1.333833	1.335722
1.60	1.518488	1.521514
2.00	1.683095	–

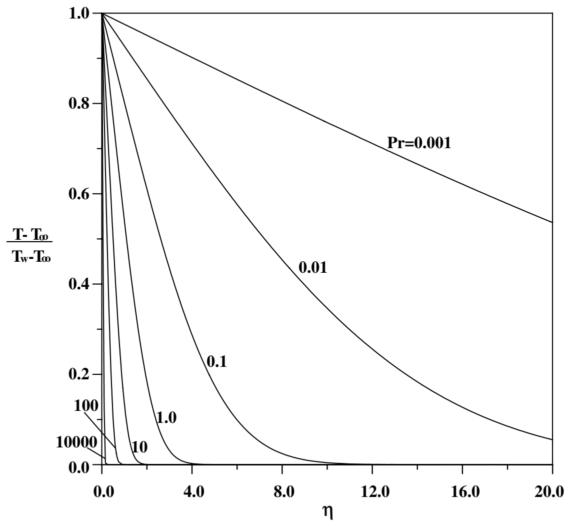

Fig. 3. Dimensionless temperature profiles for $\beta = 0$ and various Prandtl number.

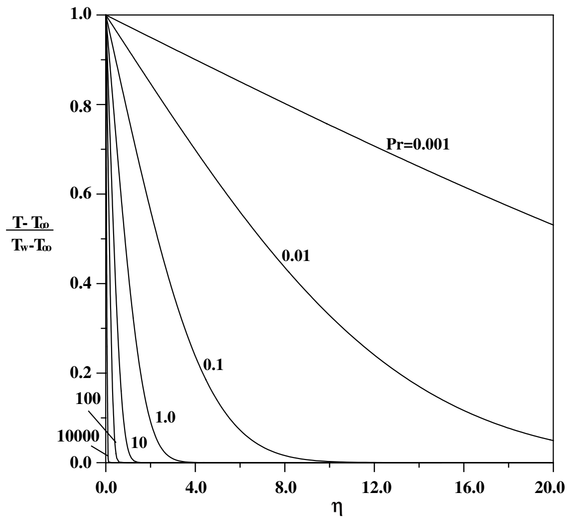

Fig. 4. Dimensionless temperature profiles for $\beta = 2$ and various Prandtl number.

results indicate that steeper velocity profiles are associated with larger values of the wedge angle parameter, β. The wedge angle parameter is a measure of the pressure gradient, and so a positive value of β indicates a negative (or favorable) pressure gradient. For accelerated flows (i.e. positive values of β), the f' profiles merely squeeze closer and closer to the wall, and overshoot or backflow phenomena are not noted. Table 2 presents a comparison of the current numerical results of $f''(0)$ for various values of β with those presented by Rajagopal et al. [3]. It is noted that there is good agreement between the two sets of results. Figs. 3 and 4 plot the dimensionless temperature distributions of the Falkner–Skan boundary-layer problem for the Prandtl number range of 0.001–10,000. Finally, Fig. 5 shows the dimensionless temperature profiles for various values of β and Prandtl number. It is noted that the maximum difference in the dimensionless temperature distributions for various values of β occurs at larger values of Prandtl number, and that this difference decreases as the Prandtl number decreases. Table 3 presents a comparison of the current numerical results of $d\theta(\eta)/d\eta$ for different values of β and Prandtl number with those presented by White [9]. Once again, it is seen

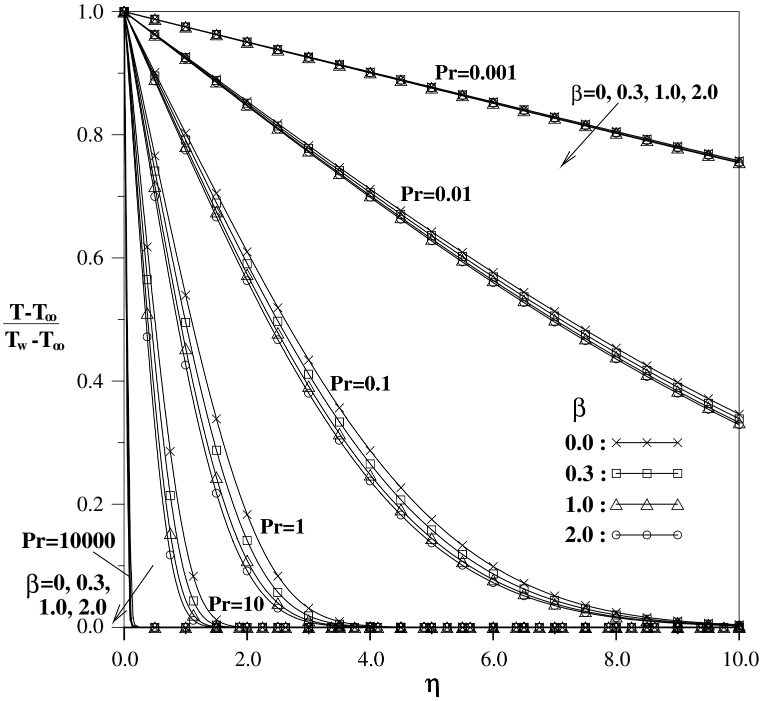

Fig. 5. Dimensionless temperature profiles for various values of β and Prandtl number.

Table 3
Numerical values of $\mathrm{d}\theta(\eta)/\mathrm{d}\eta$ for various Prandtl numbers and wedge angle parameters

Pr	$\frac{\mathrm{d}\theta(\eta)}{\mathrm{d}\eta}$							
	$\beta = 0.0$		0.3		1.0		2.0	
	White [9]	Present	White [9]	Present	White [9]	Present	White [9]	Present
0.001	0.02449	0.02449	0.02467	0.02468	0.02483	0.02483	0.02492	0.02492
0.003	0.04154	0.04154	0.04206	0.04207	0.04252	0.04253	0.04278	0.04279
0.006	0.05759	0.05760	0.05859	0.05862	0.05947	0.05949	0.05999	0.06001
0.010	0.07296	0.07296	0.07455	0.07458	0.07597	0.07599	0.07681	0.07683
0.030	0.11935	0.11935	0.12353	0.12360	0.12374	0.12740	0.12972	0.12968
0.060	0.16050	0.16050	0.16791	0.16802	0.17480	0.17488	0.17903	0.17908
0.100	0.19803	0.19803	0.20908	0.20923	0.21950	0.21962	0.22600	0.22608
0.300	0.30371	0.30372	0.32783	0.32812	0.35147	0.35168	0.36681	0.36695
0.600	0.39168	0.39168	0.42892	0.42932	0.46633	0.46661	0.49130	0.49149
0.720	0.41786	0.41809	0.45929	0.45998	0.50113	0.50174	0.52928	0.52980
1.000	0.46960	0.46960	0.51952	0.51999	0.57047	0.57080	0.60520	0.60541
2.000	0.59723	0.59723	0.66905	0.66963	0.74372	0.74412	0.79599	0.79624
3.000	0.68596	0.68596	0.77344	0.77409	0.86522	0.86565	0.93036	0.93062
6.000	0.86728	0.86728	0.98727	0.98806	1.1147	1.1152	1.2069	1.2072
10.000	1.02974	1.02974	1.1791	1.1800	1.3388	1.3394	1.4557	1.4561
30.000	1.4873	1.4873	1.7198	1.7210	1.9706	1.9714	2.1577	2.1582
60.000	1.8746	1.8746	2.1776	2.1791	2.5054	2.5063	2.7520	2.7525
100.000	2.2229	2.2229	2.5892	2.5910	2.9863	2.9874	3.2863	3.2869
400.000	3.5292	3.5292	4.1331	4.1359	4.7894	4.7910	5.2890	5.2900
1,000.000	4.7901	4.7901	5.6230	5.6268	6.5291	6.5314	7.2212	7.2225
4,000.000	7.6039	7.6039	8.9481	8.9540	10.4112	10.4147	11.5320	11.5341
10,000.000	10.3201	10.3201	12.1577	12.1657	14.1583	14.1630	15.6928	15.6956

that there is good agreement between the two sets of results.

5. Conclusions

The present paper has discussed the applicability of the differential transformation method to obtain the temperature distributions for a flow passing over a wedge. The differential transformation method which has been applied within the current study directly yields a power series, close-form solution for a system of nonlinear differential equations and requires no iterative calculations. Numerical results of the Falkner–Skan thermal boundary-layer problem have been presented in order to demonstrate the accuracy and versatility of the differential transformation method. It has been demonstrated that the obtained numerical results for the velocity and temperature distributions are in good agreement with those provided by other numerical approximation methods. In this paper, the proposed method provides an effective numerical scheme for determining the solutions of the nonlinear Falkner–Skan thermal boundary-layer problem.

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